1. Review

Recall our definitions regarding probability. We perform some sort of "experiment", such as rolling a die. An *outcome* is result of the experiment; in this case, the outcome is an integer between one and six. The *sample space* is the set of all possible outcomes; in this example, the sample space is $S = \{1, 2, 3, 4, 5, 6\}$. We may be looking for certain outcomes, say for example, we wish to roll and even number. An *event* is a subset of the sample space, say $E = \{2, 4, 6\}$.

The *cardinality* of a set is the number of things in it. The cardinality of the set A is denoted $|A|$. In our example, $|S| = 6$ and $|E| = 3$.

The *probability* of event E is defined to be

$$P(E) = \frac{|E|}{|S|}.$$

We have defined n factorial as the product of all positive integers less than or equal to n:

$$n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n.$$

This is the number of ways of rearranging n things.

2. Motivational Example

Example 1. Each card in a standard deck is identified by its *rank* and *suit*.

- Ranks: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
- Suits: $\spadesuit$, $\heartsuit$, $\diamondsuit$, $\clubsuit$

There is one card of each rank/suit combination. Since there are 13 ranks and 4 suits, there are 52 total distinct cards in a deck.

A poker hand consists of five distinct cards, in no particular order. A flush is a poker hand in which all of the cards have the same suit. Find the probability of being dealt a flush in a game of poker.

Solution Part 1. We set up the problem using a probability space.

Let D denote the set of 52 cards. A poker hand is a subset of D containing 5 cards.

The sample space S is the set of poker hands, that is, the set of subsets of D of cardinality 5:

$$S = \{H \subset D \mid |H| = 5\}.$$

Let's start by computing the probability of having a flush of hearts. The event E is the set of hands which contain only five hearts:

$$E = \{H \subset D \mid |H| = 5 \text{ and } H \text{ contains only hearts } \}.$$

We know that $P(E) = \frac{|E|}{|S|}$, but how do we count the sets S and E. □

3. Permutations and Combinations

We have a set of n things and we wish to select k distinct things from this set. We first count the number of ways of doing this when the order matters, and then when the order does not matter.

3.1. **Permutations.** A *permutation* is an ordered list of distinct elements from a given set. The number of permutations of n things taken k at a time is denoted $P(n,k)$.

To count the number of ways to select k things from a set of n things with order, we first select one for the first slot, and have n choices; now, however, we have one less choice for the second slot, so we multiply by $n-1$. There are $n-2$ choices for the third slot, and so forth. We do this k times, multiplying as we go, and see that

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!},$$

to obtain

$$P(n,k) = \frac{n!}{(n-k)!}.$$

3.2. **Combinations.** A *combination* is a subset of a given set. It is like a permutation, except that the order does not matter. The number of combinations of n things taken k at a time is denoted $C(n,k)$, or $\binom{n}{k}$. This is read "n choose k".

To count the number of ways to select k things from a set of n things without order, we use the last result. Consider an unordered sample of size k, chosen without duplicates from a set of n objects. There are $k!$ ways to arrangement this sample into an ordered list, which implies that in our above computation of order sample without replacement, each such set was counted $k!$ times. Thus, we divide the number of ordered samples without replacement by $k!$, to obtain

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Note that

$$C(n,k) = \binom{n}{k} \quad \text{and} \quad P(n,k) = \binom{n}{k}k!.$$

We now solve our motivational problem.

Example 1 (continued)**.** Find the probability of being dealt a flush in a game of poker.

Solution. Let D denote a set of cards. Then $|D| = 52$. A poker hand is a set of five cards. Thus the sample space is the set of subsets of D of cardinality five; that is,

$$S = \{H \subset D \mid |H| = 5\}.$$

We have

$$|S| = \binom{52}{5} = \frac{52!}{5!\cdot 17!} = 2598960.$$

The event E is the set of all poker hands whose cards are from one suit. There are 13 cards in a suit, so there are 13 choose 5 ways of creating a flush in a given suit. Since there are four suits, the total number of flush hands is

$$|E| = 4\cdot\binom{13}{5} = 4\cdot\frac{13!}{5!\cdot 8!} = 4\cdot 1287 = 5248.$$

Thus

$$P(E) = \frac{5248}{2598960} = 0.00198.$$

□